Decoupling of superfluid and normal pulsation modes in neutron and hyperon stars

M.E. Gusakov, E.M. Kantor

Ioffe Institute, St.-Petersburg
Outline

• Introduction and formulation of the problem

• Superfluid hydrodynamics

• Pulsation equations in the linear approximation

• Superfluid and normal modes

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• Decoupling in superfluid hyperon stars

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Introduction and formulation of the problem
• Studying the pulsations of General Relativistic neutron stars (NSs) is very important and actively developing area of research, since comparison of pulsation theory with observations can give valuable information about the properties of superdense matter.

• Yet, it is a very difficult theoretical problem even for normal (nonsuperfluid) stars.

• Additional complication: At $T \leq 10^8 - 10^{10}$ K baryons in the internal layers of NSs become superfluid
Lindblom and Mendell (1994) were the first who studied global pulsations of superfluid NSs.

Considering a simple model of a Newtonian star with nucleon core they numerically found two distinct classes of pulsation modes:

• normal-like modes, which practically coincide with the corresponding modes of a normal star

• superfluid modes in which the matter pulsates in such a way that the mass current density approximately vanishes

Subsequent numerical studies verified the result of Lindblom and Mendell though no general explanation of this result has been proposed.
In the first part of the talk we explain these numerical results for npe-matter by demonstrating that:

- Normal and superfluid modes are described by two sets of weakly coupled equations
- The coupling between them is parameterized by just ONE coupling parameter $s$ which is small for realistic equations of state

These results allow us to formulate a simple perturbative (in parameter $s$) scheme which considerably simplifies the problem of calculation of the pulsation spectrum for superfluid NSs. They also lead to a number of important physical conclusions about the properties of superfluid oscillations.

In the second part of the talk, we will demonstrate how these results can be generalized to a more general case of superfluid hyperon stars.
The main assumptions

• We first consider *npe-matter*

• Unperturbed star is in beta-equilibrium: \( \delta \mu \equiv \mu_n - \mu_p - \mu_e = 0 \)
  \( \mu_i \) is the chemical potential for particles \( i=n, p, e \)

• Quasineutrality (also for perturbed star): \( n_e = n_p \)
  \( n_i \) is the number density of particles \( i=n, p, e \)

In this talk it is additionally assumed that:

• a star is nonrotating (no Feynman-Onsager vortices)

• in the first part of the talk we also assume that protons are normal while neutrons are superfluid in some region of a NS core

These assumptions are made just to simplify the presentation and do not affect our principal results.
Superfluid hydrodynamics
The main feature of superfluid hydrodynamics is the presence of two (or more) independent velocity fields.

For normal matter:

\[
\begin{align*}
    j_{(e)}^\mu &= n_e u^\mu \\
    j_{(p)}^\mu &= n_p u^\mu \\
    j_{(n)}^\mu &= n_n u^\mu 
\end{align*}
\]

the standard expressions for particle current density

When neutrons are superfluid:

\[ j_{(n)}^\mu = n_n u^\mu + Y_{nn} w_{(n)}^\mu \]

\[ w_{(n)}^\mu \equiv \mu_n (V_{s(n)}^\mu - u^\mu) \]

2 velocity fields instead of one

\[ u^\mu \]

is the velocity of a normal component (electrons, protons, and normal neutrons)

\[ V_{s(n)}^\mu \]

is the velocity of superfluid neutrons

where the “superfluid” four-vector \( w_{(n)}^\mu \) is given by:

proportional to a difference between the superfluid and normal velocity
\[ j^{\mu}_{(n)} = n_n u^{\mu} + Y_{nn} \omega^{\mu}_{(n)} \]

The physical meaning of the coefficient \( Y_{nn} \):

\[ \mu_n Y_{nn} = \text{effective number density of superfluid neutrons} \]

\[ Y_{nn} \in [0, n_n / \mu_n] \]

(Gusakov, Kantor & Haensel 2009)
The basic equations

- Conservation of baryons and electrons
  
  \( (n_b \, U^\mu)_{;\mu} = 0 \)
  
  \( (n_e \, u^\mu)_{;\mu} = 0 \)

  \( U^\mu \equiv u^\mu + \frac{Y_{nn} w^\mu_{(n)}}{n_b} \)

- Einstein equation
  
  \( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G \, T^{\mu\nu} \)

  \( T^{\mu\nu} = (P + \varepsilon) u^\mu u^\nu + Pg^{\mu\nu} + Y_{nn}(w^\mu_{(n)} w^\nu_{(n)} + \mu_n w^\mu_{(n)} u^\nu + \mu_n w^\nu_{(n)} u^\mu) \)

- Potentiality condition for motion of superfluid neutrons
  
  \( \partial_\nu \left[ w_{(n)\mu} + \mu_n u_\mu \right] - \partial_\mu \left[ w_{(n)\nu} + \mu_n u_\nu \right] = 0 \)

- The second law of thermodynamics
  
  \( d\varepsilon = \mu_n \, dn_b - \delta \mu \, dn_e + T \, dS \)
Pulsation equations in the linear approximation
One can derive the following linearized “superfluid” equation:

\[ i \omega (\mu_n Y_{nn} - n_b) \delta w_{(n)} j = \sqrt{-g_{00}} n_e \partial_j (\delta \mu) \quad j = 1, 2, 3 \]

\[ \delta \mu = \delta \mu(\delta g^{\mu\nu}, \delta U^\mu, \delta w_{(n)}^\mu) \]

depends on the perturbations of \( \delta g^{\mu\nu}, \delta U^\mu, \) and \( \delta w_{(n)}^\mu \)

The linearized Einstein equations can be presented (very schematically) in the following symbolic form:

\[ \delta (R^{\mu\nu} - 1/2 g^{\mu\nu} R) = 8\pi G \delta T_{\text{norm}}^{\mu\nu} + s \times \left( \text{terms depending on } \delta w_{(n)}^\mu \right) \]

contains only the metric

However, it can be shown that if \( s \) is small, then one can present the solution to the system of pulsation equations in the form of a series in parameter \( s \).

So, let us inspect the coupling parameter \( s \):

\[ s \equiv \frac{n_e}{n_b} \frac{\partial P/\partial n_e}{\partial P/\partial n_b} \]

Generally, both these equations are coupled!
The coupling parameter versus density

The stiffer EOS, the smaller $s$.

Recent measurements of the mass of the pulsar PSR J1614-2230 by P. Demorest et al. (2010) indicate that EOS is stiff

$$M_{\text{max}} > 1.97M_{\odot}!$$

$$|s| \sim 0.01 - 0.05$$

approximation of vanishing $s$ should be very reasonable
Superfluid and normal modes
What will happen if $s$ vanishes?

If $s=0$ then the normal equation does not depend on $\delta \omega^{\mu}_{(n)}$ and has **exactly** the same form as in the absence of superfluidity.

$$
\delta \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 8\pi G \, \delta T^{\mu\nu}_{\text{norm}}
$$

contains only the metric perturbations $\delta g^{\mu\nu}$

$$
\delta T^{\mu\nu}_{\text{norm}} = \delta T^{\mu\nu}_{\text{norm}}(\delta g^{\mu\nu}, \delta U^{\mu})
$$

this quantity is formally given by exactly the same expression as the perturbation of $\delta T^{\mu\nu}$ for normal matter.

The solution (the spectrum of eigenfunctions $\delta U^{\mu}$ and eigenfrequencies $\omega$) is indistinguishable from that for a nonsuperfluid star.

We obtained the **normal modes**
Assume that $s$ still vanishes. Is it possible for a NS to oscillate on a frequency which is \textit{not} an eigenfrequency of a normal star?

If \textbf{yes} then:  $\delta U^\mu = 0$ and $\delta g^{\mu \nu} = 0$

\[
\delta \mu = \frac{n_e (\partial \delta \mu / \partial n_e)}{i \omega U^0} \left[ \frac{\partial_j n_e}{n_e} \frac{Y_{nn} \delta w^j_{(n)}}{n_b} + \left( \frac{Y_{nn} \delta w^j_{(n)}}{n_b} \right)_{;j} \right]
\]

depends only on $\delta w^\mu_{(n)}$

\[
i \omega \left( \mu_n Y_{nn} - n_b \right) \delta w_{(n)j} = \sqrt{-g_{00}} n_e \partial_j (\delta \mu) \quad \quad j = 1, 2, 3
\]

In this approximation the superfluid equation is self-contained and allows to determine the eigenfrequencies $\omega$ and the eigenfunctions $\delta w^\mu_{(n)}$

\textbf{We obtained the superfluid modes}
Properties of the superfluid modes in the zero approximation (s=0)

• They do not emit gravitational waves, $\delta g^{\mu\nu} = 0$

• The oscillating quantity is $\delta \mu = \mu_n - \mu_p - \mu_e$; the pressure and baryon current density are not perturbed, $\delta P = 0; \delta j^{(b)}_{\mu} = 0$

• The modes are entirely localized in the superfluid region of a star (in particular, they do not appear at the NS surface)

In the consideration above we assumed that $s=0$. Clearly, superfluid and normal modes should remain approximately decoupled also at small but finite coupling parameter $s$.

Moreover, since $|s| \sim 0.01 - 0.05$, the approximation of vanishing $s$ is already sufficient to calculate the spectrum within accuracy $\sim s$ (i.e., a few per cent).
Example: Radial pulsations
Superfluid (dashes) and normal (solid lines) modes for a radially pulsating NS. The spectrum is calculated assuming \( s=0 \).

Six superfluid modes (1,...,6) and three normal modes (I, II, III)

\[
M = 1.4M_\odot
\]

EOS APR
(Akmal et al. 1998)

\[
\omega_0 = \frac{c}{R} \quad \| \quad R = 12.17 \text{ km}
\]

Neutrons in the core are superfluid at \( T \leq T_{cn}^\infty = 6 \times 10^8 \text{ K} \).
Comparison with the exact solution to the system of coupled pulsation equations

Approximate

Exact

The spectrum is not plotted in the shaded region

Avoided crossings

Ordinary crossings

Exact solution: Gusakov and Andersson (2006)
Approximate (dashes) and exact (solid lines) spectra

Approximate solution fits very well the exact solution and differs from it by 1—3 %
Decoupling in superfluid hyperon stars
Could this approach be generalized to hyperon matter?

A few facts about hyperons and nucleon-hyperon matter:

- They first appear at densities $\sim (2 - 3)\rho_0 \approx 2.8 \times 10^{14} \text{ g cm}^{-3}$

One has to take hyperons into account

- Hyperon equation of state (EOS) is poorly known

In what follows we use two model EOSs:

The “old” $\sigma \omega \rho$ -model of Glendenning (his model III) hereafter **EOS GIII**

$M_{\text{max}} = 1.975M_\odot$

Recent highly nonlinear $\sigma \omega \rho \phi$ -model of Bednarek & Manka (2009) (hereafter **EOS MB16**)

$M_{\text{max}} = 1.988M_\odot$

Three types of hyperons for stable neutron-star configurations: $\Lambda$, $\Xi^-$, and $\Sigma^-$

For more details see the poster of I. Bednarek et al.
• Nucleon-hyperon matter out of and in thermodynamic equilibrium

**7 particle species:** \[ e, \mu, n, p, \Lambda, \Xi^-, \Sigma^- \]

Generally, any thermodynamic quantity, e.g., pressure, is a function of 7 number densities:

\[
P = P(n_e, n_\mu, n_n, n_p, n_\Lambda, n_\Xi, n_\Sigma)
\]

Not all of them are independent: **3 conditions**

Equilibrium with respect to the fast reactions \( n + \Lambda \leftrightarrow p + \Sigma^- \) and \( \Lambda + \Lambda \leftrightarrow p + \Xi^- \):

\[
\begin{align*}
\mu_n + \mu_\Lambda &= \mu_p + \mu_\Sigma \\
2\mu_\Lambda &= \mu_p + \mu_\Xi
\end{align*}
\]

because the reactions are strong, the equilibration time scale is very short comparing to pulsation period, so that these conditions are satisfied even for the star out of thermodynamic equilibrium and quasineutrality condition:

\[
n_p = n_e + n_\mu + n_\Xi + n_\Sigma
\]

only 7-3=4 independent number densities.

To decouple the superfluid and normal modes it is convenient to choose \( n_b, n_e, n_\mu, \)

and the strangeness density \( n_{\text{str}} = \sum_{i \in \text{baryons}} S_i n_i \) as the independent variables.

\( i = n, p, \Lambda, \Xi, \Sigma \)
In thermodynamic equilibrium we have 3 additional constraints for disbalance of chemical potentials:

\[
\begin{align*}
\delta \mu_e & \equiv \mu_n - \mu_p - \mu_e = 0 \\
\delta \mu_\mu & \equiv \mu_n - \mu_p - \mu_\mu = 0 \\
\delta \mu_b & \equiv \mu_n - \mu_\Lambda = 0
\end{align*}
\]

For a perturbed star the quantities $\delta \mu_e$, $\delta \mu_\mu$, and $\delta \mu_b$ are nonzero but small (linear in pulsation amplitude).
• the hyperon critical temperatures are high ($T_c \geq 10^9$ K in the maximum)

To model pulsations one should use the **superfluid** hydrodynamics

For **EOS MB16** one has

to take into account possible superfluidity of

$$n, \ p, \ \Lambda, \ \Xi^-, \ and \ \Sigma^-$$

*The hydrodynamics for superfluid nucleon-hyperon mixtures is similar to that for npe-matter, see Kantor & Gusakov PRD 79, 043004 (2009) for more details.*
Using this hydrodynamics one obtains the following superfluid equations

2 main equations:  \( i, k = n, p, \Lambda, \Xi, \Sigma \)

\[
i \omega (\mu_i Y_{ik} \delta w_{(k)j} - n_b \delta w_{(n)j}) = \sqrt{-g_{00}} \left[ n_e \partial_j (\delta \mu_e) + n_{\mu} \partial_j (\delta \mu_{\mu}) + n_{\text{str}} \partial_j (\delta \mu_b) \right]
\]
\[
i \omega (\delta w_{(\Lambda)j} - \delta w_{(n)j}) = \partial_j (\sqrt{-g_{00}} \delta \mu_b)
\]

and 3 supplementary equations independent of pulsation frequency \( \omega \) :

\[
\delta w_{(n)j} + \delta w_{(\Lambda)j} = \delta w_{(p)j} + \delta w_{(\Sigma)j} \quad \text{consequence of the requirement of equilibrium with respect to the fast reaction } n + \Lambda \leftrightarrow p + \Sigma
\]
\[
2 \delta w_{(\Lambda)j} = \delta w_{(p)j} + \delta w_{(\Xi)j} \quad \text{consequence of the requirement of equilibrium with respect to the fast reaction } \Lambda + \Lambda \leftrightarrow p + \Xi
\]
\[
(Y_{pi} - Y_{\Sigma i} - Y_{\Xi i}) \delta w_{(i)}^\mu = 0 \quad \text{consequence of charge neutrality}
\]

5 equations for 5 superfluid four-vectors:  \( \delta w_{(n)}^\mu, \delta w_{(p)}^\mu, \delta w_{(\Lambda)}^\mu, \delta w_{(\Xi)}^\mu, \delta w_{(\Sigma)}^\mu \).

\( Y_{ik} \) is the relativistic entrainment matrix, generalization of \( Y_{nn} \) to the case of superfluid mixtures [see the poster of M. Timofeeva et al. at this conference].
Following the same strategy as in the case of npe-matter, one can present the linearized Einstein equation in the form

\[ \delta(R^{\mu\nu} - 1/2 \ g^{\mu\nu} \ R) = 8\pi G \left( \delta T_{\text{norm}}^{\mu\nu} + s_l \times \sum_{i \in \text{baryons}} Y_{ik} \delta w_k^\mu \right) + s_{\text{str}} \times \left( \text{terms depending on } s_i Y_{ik} \delta w_k^\mu \right) \]

contains only the metric perturbations \( \delta g^{\mu\nu} \)

\[ \delta T_{\text{norm}}^{\mu\nu} = \delta T_{\text{norm}}^{\mu\nu}(\delta g^{\mu\nu}, \delta U^\mu) \]

coincides with \( \delta T^{\mu\nu} \) for normal matter

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**Two coupling parameters instead of one!**

\[ s_l \equiv \frac{n_e}{n_b} \frac{\partial P/\partial n_e}{\partial P/\partial n_b} + \frac{n_\mu}{n_b} \frac{\partial P/\partial n_\mu}{\partial P/\partial n_b} \]

“lepton” coupling parameter

\[ s_{\text{str}} \equiv \frac{n_{\text{str}}}{n_b} \frac{\partial P/\partial n_{\text{str}}}{\partial P/\partial n_b} \]

“strange” coupling parameter

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Again, as for npe-matter, if \( s_l = 0 \) and \( s_{\text{str}} = 0 \) then superfluid degrees of freedom are completely decoupled from metric perturbations \( \delta g^{\mu\nu} \) and the baryon four-velocity \( \delta U^\mu \)

*Let’s look, whether these coupling parameters are small or not*
Coupling parameters

\[ \sigma \omega \rho \text{-model (EOS GIII)} \]

\[ \sigma \omega \rho \sigma^* \phi \text{-model (EOS MB16)} \]

Thus, the approximation of \( s_l = 0 \) and \( s_{str} = 0 \) should be reasonable.
Let us illustrate this conclusion by considering sound waves in superfluid nucleon-hyperon matter.

We have:

2 superfluid pulsation equations → 2 superfluid sound modes

1 normal pulsation equation → 1 normal mode
Speed of sound modes versus temperature for \textit{EOS GIII}

- **Exact** solution to the system of coupled pulsation equations
- **Approximate** solution of completely decoupled superfluid and normal equations

The agreement between approximate and exact solutions for \textit{EOS MB16} should be even better, especially at high densities (because coupling parameters for this EOS are smaller)

\textit{This work is in progress!}
Conclusions
• We demonstrate that equations governing oscillations of superfluid neutron and hyperon stars can be split into two systems of weakly coupled equations. One system of equations describes normal modes, another one – superfluid modes.

• The coupling of these systems is small for realistic EOSs. We have 1 coupling parameter for neutron stars ($|s| \sim 0.01 - 0.05$) and 2 coupling parameters for hyperon stars ($|s_l|, |s_{str}| < 0.1$).

• Already an approximation of vanishing coupling parameters is sufficient to calculate the pulsation spectrum within accuracy of a few per cent.

• Pulsation spectra for normal modes can be calculated using the ordinary nonsuperfluid hydrodynamics. (However, to study dissipation of normal modes one has to take into account the specific superfluid dissipative terms! We know these terms since we can easily calculate the “superfluid” eigenfunctions using the method proposed in this talk.)

• The obtained results suggest a simple perturbative (in coupling parameters) scheme which considerably simplifies the problem of calculation of the pulsation spectrum for superfluid neutron and hyperon stars.

• The proposed approach allows to take into account realistic EOSs, temperature effects, dissipation, baryon superfluidity, density-dependent profiles of critical temperatures, and stellar rotation.
Properties of superfluid modes

• In the zero approximation (\( s = 0 \) or \( s_l, s_{\text{str}} = 0 \)) superfluid modes:
  (a) do not perturb pressure and baryon current density;
  (b) do not appear at the stellar surface
    (localized in a superfluid region of a star);
  (c) do not perturb metric (no gravitational radiation).

• These results indicate that superfluid modes should be very difficult to observe at small but finite coupling parameters. This means that observational properties of a pulsating superfluid star and a normal star of the same mass should be very similar, so that it will be very hard to discriminate one from the other.

• Gravitational radiation from the global superfluid modes is possible only in the next (first) order of perturbation theory in coupling parameters. Thus, it should be suppressed in comparison to that from normal modes by a factor of \( s^2 \sim 10^{-3} \) for neutron stars and \( s_l^2, s_{\text{str}} \sim 10^{-2} \) for hyperon stars.

For more details see: *M.E. Gusakov & E.M. Kantor, PRD 83, 081304(R) (2011)*